Math 142 Lecture 6 Notes

Daniel Raban

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1 Compactness and Analysis

1.1 Compact subsets of Hausdorff spaces

Here is the theorem we promised to prove last time.

Theorem 1.1. *If X is Hausdorff, and $A \subseteq X$ is compact, then A is closed.*

Proof. We will show that $X \setminus A$ is open. Let $x \in X \setminus A$, and choose $z \in A$. X s Hausdorff, so there exist neighborhoods U_z, V_z such that $x \in U_z$, $z \in V_z$, and $U_z \cap V_z = \varnothing$. We can vary z to get a collection $\{V_z : z \in A\}$ of open sets. Then $\{V_z \cap A\}$ is an open cover of A. A is compact, so $A = (A \cap V_{z_1}) \cup \cdots \cup (A \cap V_{z_n})$ for some $z_1, \ldots, z_n \in A$. This implies that $A \subseteq V_{z_1} \cup \cdots \cup V_{z_n} = V$.

Since $U_{z_i} \cap V_{z_i} = \varnothing$, we know that $U = U_{z_1} \cap \cdots \cap U_{z_n}$ is disjoint from V. So $U \cap A = \varnothing$; i.e. $U \subseteq X \setminus A$. Also, U is open (as an intersection of finitely many open sets), and $x \in U$, so we have an open neighborhood of x that is contained in $X \setminus A$. Since x was any point in $X \setminus A$, we conclude that $X \setminus A$ is open. Hence, A is closed. □

1.2 Generalizations of theorems from analysis

1.2.1 The Bolzano-Weiertrass theorem

Recall the following theorem from analysis.

Theorem 1.2 (Bolzano-Weierstrass). *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Given a sequence (a_n), we can construct the set $\{a_n : n \in \mathbb{N}\} \subseteq \mathbb{R}$. We can think of the limit of a sequence as a limit point of $\{a_n\}$ (if $\{a_n\}$ infinite). This gives rise to a more general topological analogue of the Bolzano-Weierstrass theorem.

Theorem 1.3 (Bolzano-Weierstrass). *If X is compact, and $A \subseteq X$ is infinite, then A has a limit point.*

Proof. In pursuit of a contradiction, assume A has no limit points; we will show that A must be finite. Given $x \in X$, it is not a limit point of A. So if $x \in A$, then we can find a neighborhood U_x of x such that $U_x \cap (A \setminus \{x\}) = \varnothing$; i.e. $U_x \cap A = \{x\}$. Likewise, if $x \notin A$, we can find a neighborhood U_x of x such that $U_x \cap (A \setminus \{x\}) = \varnothing$; i.e. $U_x \cap A = \varnothing$. Then $\{U_x\}$ is an open cover of X, so $X = U_{x_1} \cup \cdots \cup U_{x_n}$, as X is compact. But $A = A \cap X = (A \cap U_{x_1}) \cup \cdots (A \cap U_{x_n})$. We had that $|A \cap U_{x_i}| \leq 1$ for each i, so $|A| \leq n < \infty$. This is a contradiction, as A was assumed to be infinite. $\square$

1.2.2 Characterization of compactness

Theorem 1.4. *If $A \subseteq \mathbb{R}^n$ is compact, then it is closed and bounded.*

Proof. $\mathbb{R}^n$ is Hausdorff, so since A is a compact subset, it is closed. Since $A \subseteq \bigcup_{n=1}^{\infty} B_n(0)$, $\{A \cap B_n(0)\}$ is an open cover of A. A is compact, so $A = (A \cap B_{n_1}(0)) \cup \cdots \cup (A \cap B_{n_k}(0))$. Take $N = \max\{n_1, \dots, n_k\}$. Then $A = A \cap B_N(0)$; i.e. $A \subseteq B_N(0)$, so it is bounded. $\square$

Recall the following theorem from analysis.

Theorem 1.5. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

In the general topological setting, this becomes the following theorem.

Theorem 1.6. *If X is compact, and $f : X \to \mathbb{R}$ is continuous, then f is bounded and attains its bounds.*

Proof. The image $f(X)$ is compact, so $f(X)$ is closed and bounded by the theorem we just proved. Since $f(X)$ is bounded and nonempty, it has a supremum S and an infimum I. We know that S and I are limit points of $f(X)$ (if $f(X)$ is finite, the supremum is just one of the points). The set $f(X)$ is closed, so it contains its limit points. So $S, I \in f(X)$; i.e. $S = f(x_0)$ and $I = f(x_1)$ for some $x_0, x_1 \in X$, so f attains its bounds. $\square$

1.3 Tychonoff's product theorem (finite version)

We want to prove the converse to the previous theorem that sats compact $\implies$ clsoed and bounded in $\mathbb{R}^n$. To do that, we will establish a more general theorem about compactness of product spaces. First, we need a lemma.

Lemma 1.1. *If $\{U_i\}$ is a base for the topology of a space X, then X is compact iff every open cover $\mathcal{C}$ of X such that $\mathcal{C} \subseteq \{U_i\}$ has a finite subcover.*

Proof. ($\implies$) This follows from the definition of compactness.

($\impliedby$) Let $\mathcal{C}$ be any open cover of X, and let $\mathcal{B}$ be a base. We build a new open cover $\mathbb{C}'$. For each $A \in \mathcal{C}$, $A = \bigcup_i U_i$, where $U_i \in \mathcal{B}$. Let $\mathcal{C}' := \{U_i \in \mathcal{B} : \exists A \in \mathcal{C} \text{ such that } U_i \subseteq A\}$. By assumption, $\mathcal{C}'$ has a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$. For each $i = 1, \dots, n$, $U_i \subseteq A_i$ for some $A_i \in \mathcal{C}$, so $X = \bigcup_{i=1}^n U_i \subseteq \bigcup_{i=1}^n A_i \subseteq X$. So $\bigcup_{i=1}^n A_i = X$ and $\mathcal{C}$ has a finite subcover. $\square$

Theorem 1.7 (Tychonoff (finite version))**.** *$X \times Y$ is compact iff X and Y are compact.*

Proof. ($\implies$) We have continuous functions $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ that are surjective. So $X = p_1(X \times Y)$ and $Y = p_2(X \times Y)$ are compact.

($\impliedby$) Let $\mathcal{C} = \{U_i \times V_i\}$ be an open cover of $X \times Y$ by open sets in the base of the product topology from the definition. We will show that $\mathcal{C}$ has a finite subcover, and then we will use the lemma.

If $x \in X$, then $p_2|_{\{x\} \times Y} : \{x\} \times Y \to Y$ is a homeomorphism. Since Y is compact, so is $\{x\} \times Y$. So there exists a subcover $\mathcal{C}_x \subseteq \mathcal{C}$ such that $\mathcal{C}_x = \{U_1^x \times V_1^x, \dots, U_{n_x}^x \times V_{n_x}^x\}$ is finite, and $\{x\} \times Y \subseteq \bigcup_{i=1}^{n_x} U_i^x \times V_i^x$.

If $U^x = U_1^x \cap \dots \cap U_{n_x}^x$, then $U^x \times Y \subseteq \bigcup_{i=1}^{n_x} U_i^x \times V_i^x$. So for every $x \in X$, we get an open set $U^x \subseteq X$; this makes $\{U^x : x \in X\}$ an open cover of X. X is compact, so $X = U^{x_1} \cup \dots \cup U^{x_s}$. Then

$$X \times Y = \bigcup_{j=1}^{s} U^{x_j} \times Y = \bigcup_{j=1}^{s} \bigcup_{i=1}^{n_{x_j}} U_i^{x_j} \times V_i^{x_j}.$$

This is a finite union, so $\mathcal{C}$ has a finite subcover. $\square$